IMPORTANT CALENDAR CHANGES

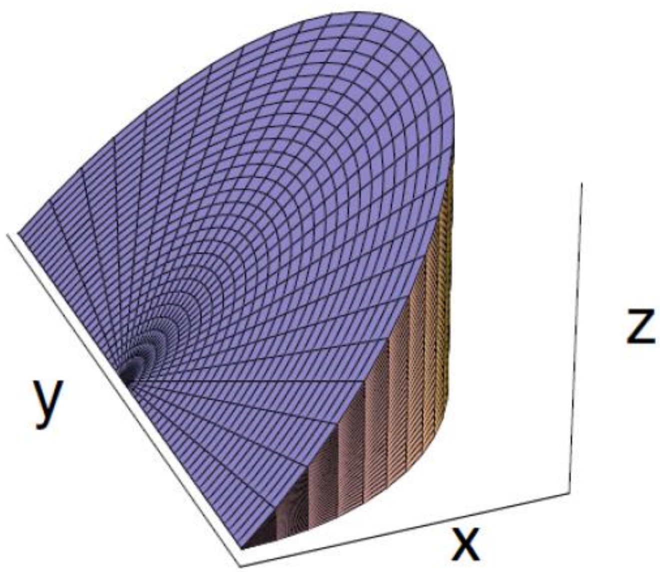
Close *Tuesday*: 15.3, 15.4

Exam 2 is THURSDAY!!!

10.3,13.4,14.1,14.3,14.4,14.7,15.1-4

Entry Task (Old Exam Question)

Find the volume of the wedge shaped solid that lies above the xy-plane, below the plane $z = x$, and within the solid cylinder $x^2 + y^2 \leq 9$.



15.4 Center of Mass

Motivation “the see-saw”

New App: Consider a thin plate (*lamina*) with density at each point given by

$\rho(x, y)$ = mass/area (kg/m^2).

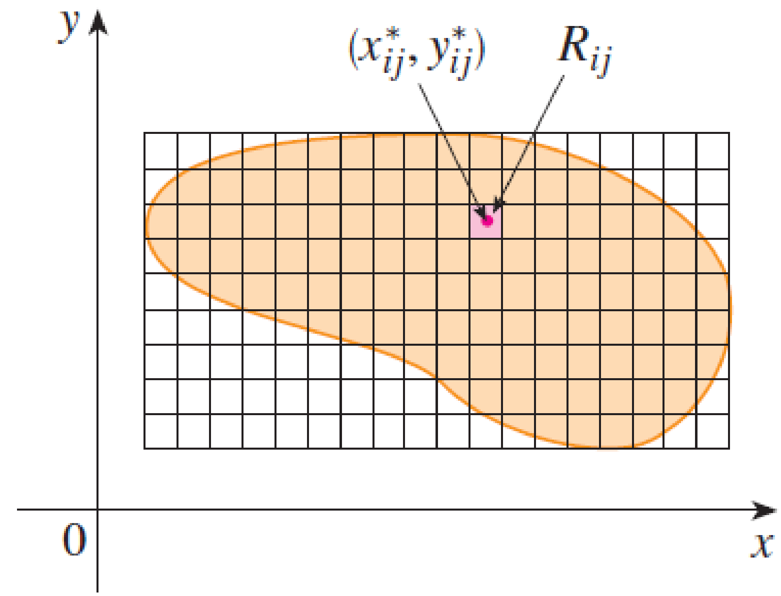
We will see that the center of mass (centroid) is given by

$$\bar{x} = \frac{\text{"Moment about y"}}{\text{Total Mass}} = \frac{\iint_R x\, p(x,y)\, dA}{\iint_R p(x,y)\, dA}$$

$$\bar{y} = \frac{\text{"Moment about x"}}{\text{Total Mass}} = \frac{\iint_R y\, p(x,y)\, dA}{\iint_R p(x,y)\, dA}$$

In general: If you are given ***n* points** $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ with corresponding masses $m_1, m_2, \ldots, m_n$ then

$$\bar{x} = \frac{m_1x_1 + \cdots + m_nx_n}{m_1 + \cdots + m_n} = \frac{M_y}{M}$$
$$\bar{y} = \frac{m_1y_1 + \cdots + m_ny_n}{m_1 + \cdots + m_n} = \frac{M_x}{M}$$



Derivation:

1. Break region into m rows and n columns.
2. Find center of mass of each rectangle:
$$(\bar{x}_{ij}, \bar{y}_{ij})$$
3. Estimate the mass of each rectangle:
$$m_{ij} = p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A$$
4. Now use the formula for *n* points.
5. Take the limit.

$$\bar{x} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}x_{ij}}{\sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}}$$
$$= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} \bar{x}_{ij}\, p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}{\sum_{i=1}^{m}\sum_{j=1}^{n} p(\bar{x}_{ij}, \bar{y}_{ij})\Delta A}$$

Center of Mass:

$$\bar{x} = \frac{\text{Moment about y}}{\text{Total Mass}} = \frac{\iint_R \; x\, p(x,y)dA}{\iint_R \; p(x,y)dA}$$

$$\bar{y} = \frac{\text{Moment about x}}{\text{Total Mass}} = \frac{\iint_R \; y\, p(x,y)dA}{\iint_R \; p(x,y)dA}$$

Example:

Consider a 1 by 1 m square metal plate. The density is given by $p(x,y) = kx$ kg/m^2 for some constant k.

Find the center of mass.

Side note:

The density $p(x,y) = kx$ means that the density is proportional to x which can be thought of as distance from the y-axis. In other words, the plate gets heavier at a constant rate from left-to-right.

Translations:

Density proportional to the dist. from…

…the y-axis -- $p(x, y) = kx$.

…the x-axis -- $p(x, y) = ky$.

…the origin -- $p(x, y) = k\sqrt{x^2 + y^2}$.

Density proportional to the <u>square</u> of the distance from the origin:

$$p(x, y) = k(x^2 + y^2).$$

Density <u>inversely</u> proportional to the distance from the origin:

$$p(x, y) = \frac{k}{\sqrt{x^2 + y^2}}$$

Example (Old Exam Question)
A lamina occupies the region *R* in the first quadrant that is above the line $y = x$ and between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
The density is proportional to the distance from the origin.
Find the *y*-coordinate of the center of mass.